

Hyperbolic quantum mechanics

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Abstract

We start to develop the quantization formalism in a hyperbolic Hilbert space. Generalizing Born's probability interpretation, we found that unitary transformations in such a Hilbert space represent a new class of transformations of probabilities which describe a kind of *hyperbolic interference*. The most interesting problem which was generated by our investigation is to find experimental evidence of hyperbolic interference. The hyperbolic quantum formalism can also be interesting as a new theory of probabilistic waves that can be developed parallelly to the standard quantum theory. Comparative analysis of these two wave theories could be useful for understanding of the role of various structures of the standard quantum formalism. In particular, one of distinguishing feature of the hyperbolic quantum formalism is the restricted validity of the superposition principle.

1 Introduction

We develop a formalism that might be called *hyperbolic quantum formalism*. Instead of the system of complex numbers \mathbf{C} , we use the system of so called hyperbolic numbers¹ \mathbf{G} (see, for example, [1], p. 21); 'physical states' are

¹We remark that the hyperbolic arithmetics is a straightforward generalization of the complex arithmetics. Therefore this paper is quite simple from the mathematical view-

represented by vectors in a \mathbf{G} -Hilbert space. Generalizing Born's probability interpretation, we found that \mathbf{G} -linear unitary transformations represent a new class of transformations of probabilities which describe a kind of *hyperbolic interference*:

$$\mathbf{P}' = \mathbf{P}_1 + \mathbf{P}_2 \pm 2\sqrt{\mathbf{P}_1\mathbf{P}_2} \cosh \theta . \quad (1)$$

The present formalism is nothing than a theory of *hyperbolic waves* of probability (compare with \mathbf{C} -quantum formalism, a theory of trigonometric waves of probability).

The most interesting problem which was generated by this investigation is to find hyperbolic interference in experiments with elementary particles (or macro systems). It might be that such results were already recoded in some experiments with elementary particles. However, they were not interpreted as an evidence of hyperbolic interference.

On the other hand, the hyperbolic quantum formalism can also be interesting as a new theory of *probabilistic waves* that can be developed parallelly to the standard quantum theory. Comparative analysis of these two wave theories could be useful for understanding of the role of various structures of the standard quantum formalism, compare with [2]. In particular, we reconsider the role of complex numbers in quantum theory from the purely probabilistic viewpoint. It seems that complex numbers were introduced into the quantum formalism to linearize the quantum probabilistic transformation:

$$\mathbf{P}' = \mathbf{P}_1 + \mathbf{P}_2 + 2\sqrt{\mathbf{P}_1\mathbf{P}_2} \cos \theta . \quad (2)$$

The linearization is performed by the \mathbf{C} -representation of real probabilities (2) on the basis of the formula:

$$A + B + 2\sqrt{AB} \cos \theta = |\sqrt{A} + \sqrt{B}e^{i\theta}|^2 . \quad (3)$$

In the same way, to linearize probabilistic transformation (1) we have to use hyperbolic amplitudes:

$$A + B \pm 2\sqrt{AB} \cosh \theta = |\sqrt{A} \pm \sqrt{B}e^{j\theta}|^2 . \quad (4)$$

Here j is the generator of the algebra \mathbf{G} of hyperbolic numbers: $j^2 = 1$.

point. We hope that it could be interesting researches working in theoretical and experimental physics.

Other distinguishing feature of the hyperbolic quantum formalism is the *restricted validity of the superposition principle*. A given state could not be decomposed with respect to an arbitrary complete system of other states. Of course, as in the complex case, we can always expand a vector with respect to a basis². However, this operation (which is well defined from the mathematical point of view) is not always meaningful from the physical point of view. Different representations are not equivalent in the hyperbolic quantum theory.

The present note is just the first step in the development of the hyperbolic quantum formalism. It would be interesting to develop this formalism as an alternative to the standard quantum formalism. However, the most interesting problem is to find the place of hyperbolic waves of probability in experimental physics.

The main ideas on hyperbolic quantum theory were presented in author's talk at the International Conference "Foundations of Probability and Physics", Växjö, Sweden-2000 (see also [3]). The author had numerous discussions on the origin of the quantum transformation of probabilities with J. Summhammer (see also [4]). I would like to thank him for these conversations.

2 Hyperbolic quantum formalism

1. Hyperbolic algebra. A hyperbolic algebra \mathbf{G} is a two dimensional real algebra with basis $e_0 = 1$ and $e_1 = j$, where $j^2 = 1$. Elements of \mathbf{G} have the form $z = x + jy$, $x, y \in \mathbf{R}$. We have $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$ and $z_1 z_2 = (x_1 x_2 + y_1 y_2) + j(x_1 y_2 + x_2 y_1)$. This algebra is commutative. We introduce an involution in \mathbf{G} by setting $\bar{z} = x - jy$. We set $|z|^2 = z\bar{z} = x^2 - y^2$. We remark that $|z| = \sqrt{x^2 - y^2}$ is not well defined for an arbitrary $z \in \mathbf{G}$. We set $\mathbf{G}_+ = \{z \in \mathbf{G} : |z|^2 \geq 0\}$. We remark that \mathbf{G}_+ is the multiplicative semigroup: $z_1, z_2 \in \mathbf{G}^+ \rightarrow z = z_1 z_2 \in \mathbf{G}_+$. It is a consequence of the equality $|z_1 z_2|^2 = |z_1|^2 |z_2|^2$.

Thus, for $z_1, z_2 \in \mathbf{G}_+$, we have $|z_1 z_2| = |z_1| |z_2|$. We introduce $e^{j\theta} = \cosh \theta + j \sinh \theta$, $\theta \in \mathbf{R}$.

We remark that

²There is some difficulty, because the system of hyperbolic numbers \mathbf{G} is only a commutative algebra and not a number field. However, it is not so important, see [1] on the general theory of Hilbert modules over commutative and noncommutative algebras.

$$e^{j\theta_1}e^{j\theta_2} = e^{j(\theta_1+\theta_2)}, \overline{e^{j\theta}} = e^{-j\theta}, |e^{j\theta}|^2 = \cosh^2 \theta - \sinh^2 \theta = 1.$$

Hence, $z = \pm e^{j\theta}$ always belongs to \mathbf{G}_+ . We also have

$$\cosh \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \quad \sinh \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}.$$

Let $|z|^2 = x^2 - y^2 > 0$. We have

$$z = |z| \left(\frac{x}{|z|} + j \frac{y}{|z|} \right) = \text{sign } x |z| \left(\frac{x \text{sign } x}{|z|} + j \frac{y \text{sign } x}{|z|} \right).$$

As $\frac{x^2}{|z|^2} - \frac{y^2}{|z|^2} = 1$, we can represent $x \text{ sign } x = \cosh \theta$ and $y \text{ sign } x = \sinh \theta$, where the phase θ is unequally defined. We can represent each $z \in \mathbf{G}_+$ as $z = \text{sign } x |z| e^{j\theta}$.

By using this representation we can easily prove that $\mathbf{G}_+^* = \{z \in \mathbf{G}_+ : |z|^2 > 0\}$ is the multiplicative group. Here $\frac{1}{z} = \frac{\text{sign } x}{|z|} e^{-j\theta}$. The unit circle in \mathbf{G} is defined as $S_1 = \{z \in \mathbf{G} : |z|^2 = 1\} = \{z = \pm e^{j\theta}, \theta \in (-\infty, +\infty)\}$. It is a multiplicative subgroup of \mathbf{G}_+^* .

2. Hyperbolic Hilbert space is \mathbf{G} -linear space (module) \mathbf{E} with a \mathbf{G} -linear product: a map $(\cdot, \cdot) : \mathbf{E} \times \mathbf{E} \rightarrow \mathbf{G}$ that is

- 1) linear with respect to the first argument:
 $(az + bw, u) = a(z, u) + b(w, u), a, b \in \mathbf{G}, z, w, u \in \mathbf{E};$
- 2) symmetric: $(z, u) = \overline{(u, z)};$
- 3) nondegenerated: $(z, u) = 0$ for all $u \in \mathbf{E}$ iff $z = 0$.

Remark. If we consider \mathbf{E} as just a \mathbf{R} -linear space, then (\cdot, \cdot) is a bilinear form which is not positively defined. In particular, in the two dimensional case we have the signature: $(+, -, +, -)$.

2. Linear space representation of states. As in the ordinary quantum formalism, we represent physical states by normalized vectors of the hyperbolic Hilbert space: $\varphi \in \mathbf{E}$ and $(\varphi, \varphi) = 1$. We shall consider only dichotomic physical variables and quantum states belonging to the two dimensional Hilbert space. So everywhere below \mathbf{E} denotes the two dimensional space. Let $a = a_1, a_2$ and $b = b_1, b_2$ be two dichotomic physical variables. We represent them by \mathbf{G} -linear operators: $|a_1 \rangle \langle a_1| + |a_2 \rangle \langle a_2|$ and $|b_1 \rangle \langle b_1| + |b_2 \rangle \langle b_2|$, where $\{|a_i \rangle\}_{i=1,2}$ and $\{|b_i \rangle\}_{i=1,2}$ are two orthonormal bases in \mathbf{E} .

Let φ be a state (normalized vector belonging to \mathbf{E}). We can perform the following operation (which is well defined from the mathematical point of view). We expand the vector φ with respect to the basis $\{|b_i \rangle\}_{i=1,2}$:

$$\varphi = \beta_1 |b_1 \rangle + \beta_2 |b_2 \rangle, \quad (5)$$

where the coefficients (coordinates) β_i belong to \mathbf{G} . As the basis $\{|b_i \rangle\}_{i=1,2}$

is orthonormal, we get (as in the complex case) that:

$$|\beta_1|^2 + |\beta_2|^2 = 1 . \quad (6)$$

However, we could not automatically use Born's probabilistic interpretation for normalized vectors in the hyperbolic Hilbert space: it may be that $\beta_i \notin \mathbf{G}_+$ (in fact, in the complex case we have $\mathbf{C} = \mathbf{C}_+$). We say that a state φ is *decomposable* with respect to the system of states $\{|b_i\rangle\}_{i=1,2}$ if

$$\beta_i \in \mathbf{G}_+ . \quad (7)$$

In such a case we can use Born's probabilistic interpretation of vectors in a hyperbolic Hilbert space:

Numbers $\mathbf{q}_i = |\beta_i|^2, i = 1, 2$, are interpreted as probabilities for values $b = b_i$ for the \mathbf{G} -quantum state φ .

We now repeat these considerations for each state $|b_i\rangle$ by using the basis $\{|a_k\rangle\}_{k=1,2}$. We suppose that each $|b_i\rangle$ is decomposable with respect to the system of states $\{|a_i\rangle\}_{i=1,2}$. We have:

$$|b_1\rangle = \beta_{11}|a_1\rangle + \beta_{12}|a_2\rangle, \quad |b_2\rangle = \beta_{21}|a_1\rangle + \beta_{22}|a_2\rangle, \quad (8)$$

where the coefficients β_{ik} belong to \mathbf{G}_+ . We have automatically:

$$|\beta_{11}|^2 + |\beta_{12}|^2 = 1, \quad |\beta_{21}|^2 + |\beta_{22}|^2 = 1 . \quad (9)$$

We can use the probabilistic interpretation of numbers $\mathbf{p}_{11} = |\beta_{11}|^2, \mathbf{p}_{12} = |\beta_{12}|^2$ and $\mathbf{p}_{21} = |\beta_{21}|^2, \mathbf{p}_{22} = |\beta_{22}|^2$.

Remark. Let us consider matrices $B = (\beta_{ik})$ and $P = (\mathbf{p}_{ik})$. As in the complex case, the matrix B is unitary: vectors $u_1 = (\beta_{11}, \beta_{12})$ and $u_2 = (\beta_{21}, \beta_{22})$ are orthonormal. The matrix P is double stochastic, namely: $\mathbf{p}_{11} + \mathbf{p}_{12} = 1, \mathbf{p}_{21} + \mathbf{p}_{22} = 1$ and $\mathbf{p}_{11} + \mathbf{p}_{21} = 1, \mathbf{p}_{12} + \mathbf{p}_{22} = 1$.

By using the \mathbf{G} -linear space calculation (the change of the basis) we get $\varphi = \alpha_1|a_1\rangle + \alpha_2|a_2\rangle$, where $\alpha_1 = \beta_1\beta_{11} + \beta_2\beta_{21}$ and $\alpha_2 = \beta_1\beta_{12} + \beta_2\beta_{22}$.

We remark that decomposability is not transitive. In principle φ may be not decomposable with respect to $\{|a_i\rangle\}_{i=1,2}$, despite the decomposability of φ with respect to $\{|b_i\rangle\}_{i=1,2}$ and the decomposability of the latter system with respect to $\{|a_i\rangle\}_{i=1,2}$.

Suppose that φ is decomposable with respect to $\{|a_i\rangle\}_{i=1,2}$. Therefore coefficients $\mathbf{p}_k = |\alpha_k|^2$ can be interpreted as probabilities for $a = a_k$ for the \mathbf{G} -quantum state φ .

As numbers β_i, β_{ik} belong to \mathbf{G}_+ , we can uniquely represent them as $\beta_i = \pm\sqrt{q_i}e^{j\xi_i}, \beta_{ik} = \pm\sqrt{p_{ik}}e^{j\gamma_{ik}}, i, k, = 1, 2$.

We find that

$$\mathbf{p}_1 = \mathbf{q}_1\mathbf{p}_{11} + \mathbf{q}_2\mathbf{p}_{21} + 2\epsilon_1\sqrt{\mathbf{q}_1\mathbf{p}_{11}\mathbf{q}_2\mathbf{p}_{21}} \cosh \theta_1, \quad (10)$$

$$\mathbf{p}_2 = \mathbf{q}_1\mathbf{p}_{12} + \mathbf{q}_2\mathbf{p}_{22} + 2\epsilon_2\sqrt{\mathbf{q}_1\mathbf{p}_{12}\mathbf{q}_2\mathbf{p}_{22}} \cosh \theta_2, \quad (11)$$

where $\theta_i = \eta + \gamma_i$ and $\eta = \xi_1 - \xi_2, \gamma_1 = \gamma_{11} - \gamma_{21}, \gamma_2 = \gamma_{12} - \gamma_{22}$ and $\epsilon_i = \pm$. To find the right relation between signs of the last terms in equations (10), (11), we use the normalization condition

$$|\alpha_1|^2 + |\alpha_2|^2 = 1 \quad (12)$$

(which is a consequence of the normalization of φ and orthonormality of the system $\{|a_i\rangle\}_{i=1,2}$). It is equivalent to the equation:

$$\sqrt{p_{12}p_{22}} \cosh \theta_2 \pm \sqrt{p_{11}p_{21}} \cosh \theta_1 = 0. \quad (13)$$

Thus we have to choose opposite signs in equations (10), (11). Unitarity of B also imply that $\theta_1 - \theta_2 = 0$. We recall that in the ordinary quantum mechanics we have similar conditions, but trigonometric functions are used instead of hyperbolic and phases γ_1 and γ_2 are such that $\gamma_1 - \gamma_2 = \pi$.

Finally, we get that (unitary) linear transformations in the \mathbf{G} -Hilbert space (in the domain of decomposable states) represent the following transformation of probabilities:

$$\mathbf{p}_1 = \mathbf{q}_1\mathbf{p}_{11} + \mathbf{q}_2\mathbf{p}_{21} \pm 2\sqrt{\mathbf{q}_1\mathbf{p}_{11}\mathbf{q}_2\mathbf{p}_{21}} \cosh \theta, \quad (14)$$

$$\mathbf{p}_2 = \mathbf{q}_1\mathbf{p}_{12} + \mathbf{q}_2\mathbf{p}_{22} \mp 2\sqrt{\mathbf{q}_1\mathbf{p}_{12}\mathbf{q}_2\mathbf{p}_{22}} \cosh \theta. \quad (15)$$

This is a kind of hyperbolic interference.³

Remark. In fact, we first derived the probabilistic transformation (14), (15) by classifying possible probabilistic transformations induced by perturbation effects [3]. Then we found the corresponding linear space representation starting with the analogy between (3) and (4).

Remark. There can be some connection with quantization in Hilbert spaces with indefinite metric as well as the theory of relativity. However, at

³By changing hyperbolic functions to trigonometric we obtain the standard quantum interference of alternatives.

the moment we cannot say anything definite. It seems that by using Lorentz-‘rotations’ we can produce hyperbolic interference in a similar way as we produce the standard trigonometric interference by using ordinary rotations.

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